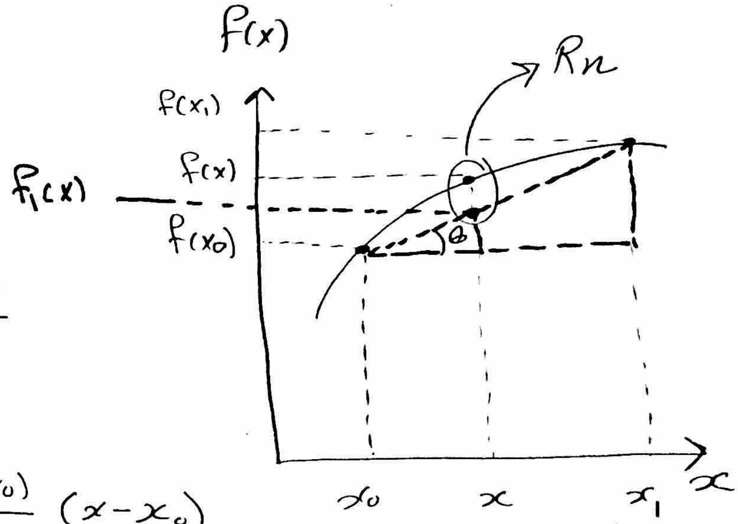
Chapter 18: Interpolation

$f(x)$: True value

$f_1(x)$: Approximate value



* Linear Interpolation

$$\tan\theta = \frac{f_1(x) - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\Rightarrow f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

If $b_0 = f(x_0)$, $b_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$ → 1st Finite-divided difference

F.D.D - $f[x_1, x_0]$

$$\Rightarrow f_1(x) = b_0 + b_1(x - x_0)$$

, $f[x_1, x_0] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

* Second order (Quadratic) Interpolation

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_1)(x - x_0)$$

$b_0 = f(x_0)$, $b_1 = f[x_1, x_0]$

$$b_2 = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = f[x_2, x_1, x_0]$$

→ 2nd F.D.D

* Nth order Interpolation

$$f_n(x) = b_0 + b_1(x - x_0) + b_2(x - x_1)(x - x_0) + \cdots +$$

$$b_n(x - x_{n-1})(x - x_{n-2}) \cdots (x - x_1)(x - x_0)$$

$$b_n = \frac{\dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} - \cdots - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_n - x_0}$$

$$= f[x_n, x_{n-1}, \ldots, x_1, x_0]$$

← Nth .F.D.D

In general

$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j}$$

$$f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k}$$

$$= \frac{\dfrac{f(x_i) - f(x_j)}{x_i - x_j} - \dfrac{f(x_j, x_k)}{x_j - x_k}}{x_i - x_k}$$

$$f[x_n, x_{n-1}, \dots, x_1, x_0] = \frac{f[x_n, x_{n-1}] - f[x_{n-1}, x_{n-2}] - \dots - f[x_1, x_0]}{x_n - x_0}$$

If we remember Taylor series

$$\text{Remainder } (R_n) = \frac{f^{(n+1)}(x)}{(n+1)!} h^{n+1}$$

Interpolation

$$\text{Remainder } (R_n) = f[x_{n+1}, x_n, \dots, x_1, x_0](x - x_n)(x - x_{n-1}) \cdots (x - x_0)$$

Error = true − approximate

Example

x_0	x_1	x_2	$f(x_0)$	$f(x_1)$	$f(x_2)$
0	1	2	-1	0	3

Find ① $f_2(1.5)$

② R_1 , $x=1.5$

solution

$$f_2(x) = b_0 + b_1(x-x_0) + b_2(x-x_1)(x-x_0)$$

$$= f(x_0) + f[x_1,x_0](x-x_0) + f[x_2,x_1,x_0](x-x_1)(x-x_0)$$

$$= -1 + (1)(1.5-0) + (1)(1.5-1)(1.5-0) = 1.25$$

$$f[x_1,x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{0-(-1)}{1-0} = 1$$

$$f[x_2,x_1,x_0] = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0}$$

$$= \frac{\frac{3-0}{2-1} - 1}{2-0} = 1$$

$$R_n = f[x_{n+1}, x_n, \dots, x_1, x_0](x-x_n)\cdots(x-x_0)$$

$n=1$

$$R_1 = f[x_2, x_1, x_0](x-x_1)(x-x_0)$$

$$= (1)(1.5-1)(1.5-0) = 0.75$$

* Lagrange Polynomials

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$ "Lagrange functions"

n = 1 (1st order)

$$f_1(x) = \sum_{i=0}^{n=1} L_i(x) f(x_i) = L_0(x) f(x_0) + L_1(x) f(x_1)$$

we need $L_0(x)$ and $L_1(x)$, $L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$

$$(i=0)\ L_0(x) = \prod_{\substack{j=0 \\ j\neq 0}}^{n=1} \frac{x - x_j}{x_i - x_j} = \frac{x - x_1}{x_0 - x_1}$$

$$(i=1)\ L_1(x) = \prod_{\substack{j=0 \\ j\neq 1}}^{n=1} \frac{x - x_j}{x_i - x_j} = \frac{x - x_0}{x_1 - x_0}$$

$$f_1(x) = \left(\frac{x - x_1}{x_0 - x_1}\right) f(x_0) + \left(\frac{x - x_0}{x_1 - x_0}\right) f(x_1)$$

$n = 2$ (2nd order)

$$f_2(x) = \sum_{i=0}^{n=2} L_i(x) f(x_i) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2)$$

we need $L_0(x), L_1(x), L_2(x)$, $\quad L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$

$$(i=0),\ L_0(x) = \prod_{\substack{j=0 \\ j \neq 0}}^{n=2} \frac{x - x_j}{x_i - x_j} = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)$$

$$(i=1),\ L_1(x) = \prod_{\substack{j=0 \\ j \neq 1}}^{n=2} \frac{x - x_j}{x_i - x_j} = \left(\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{x - x_0}{x_1 - x_0}\right)$$

$$(i=2),\ L_2(x) = \prod_{\substack{j=0 \\ j \neq 2}}^{n=2} \frac{x - x_j}{x_i - x_j} = \left(\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{x - x_0}{x_2 - x_0}\right)$$

$$\Rightarrow f_2(x) = \left(\frac{x - x_2}{x_0 - x_2}\right)\left(\frac{x - x_1}{x_0 - x_1}\right) f(x_0) + \left(\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{x - x_0}{x_1 - x_0}\right) f(x_1)$$

$$+ \left(\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{x - x_0}{x_2 - x_0}\right) f(x_2)$$

Example:

x_0	x_1	x_2	$f(x_0)$	$f(x_1)$	$f(x_2)$
1	4	6	0	1.39	1.79

Using Lagrange polynomials

Find ① $f_1(2)$, ② $f_2(2)$

Solution $(x=2)$

$$f_1(2) = \frac{2-4}{1-4}(0) + \frac{2-1}{4-1}(1.39) = 0.46$$

$$f_2(2) = \frac{(2-4)(2-6)}{(1-4)(1-6)}(0) + \frac{(2-1)(2-6)}{(4-1)(4-6)}(1.39)$$

$$+ \frac{(2-1)(2-4)}{(6-1)(6-4)}(1.79) = 0.565$$

End of chapter 18